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LIMIT ANALYSIS PROBLEM AND ITS PENALIZATION

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Abstract. The contribution is focused on solution of the kinematic limit analysis problem within associative perfect plasticity. It is a constrained minimization problem describing a collapse state of an investigated body. Two different penalization methods are presented and interpreted as the truncation method and the indirect incremental method, respectively. It is shown that both methods are meaningful even within the continuous setting of the problem. Convergence with respect to penalty and discretization parameters is discussed. The indirect incremental method can be simply implemented within current elastoplastic codes.

1 Introduction

Existence of the limit load is a feature of elastic-perfectly problems. It is well-known that an investigated body collapses when the limit value of a load parameter is exceeded [4, 16]. Strip-footing collapse or slope stability are traditional geotechnical applications, where the limit load analysis is important (see, e.g., [3, 5, 10]). We focus only on associative perfect plasticity, although the limit analysis is also meaningful for nonassociative elastoplastic models with internal variables [10, 17].

The collapse state can be described by a special variational problem, the so-called *limit analysis problem* [4, 16, 17]. It leads to a minimization of a convex functional subject to various constraints, when the kinematic approach is considered. The constraints depend on a prescribed yield function and they can cause locking phenomena. Therefore, mixed finite elements are often used for solution of such problems [1, 4].

In order to suppress the constraints, we introduce two different penalization methods for the kinematic limit analysis problem. These methods can be interpreted as the truncation method and the indirect incremental method, respectively. Both methods have been analyzed in recent papers [2, 7, 8, 15, 14], can be used for various yield criteria and lead to simple numerical techniques. Here, we recapitulate main results of these papers and slightly generalize the indirect method of incremental limit analysis.

The rest of the contribution is organized as follows. In Section 2, evolution variational formulations of the associative elastic-perfectly plastic problem with an abstract yield criterion are summarized. The implicit Euler discretization of the problem is introduced in Section 3. In Section 4, the static and kinematic limit analysis problems are formulated. In Section 5, the truncation and indirect incremental methods are derived by penalization of the kinematic limit analysis problem. The finite element approximation is discussed. In Section 6, the indirect method of incremental limit analysis is combined with Newton-like method.

2 Associative elastic-perfectly plastic problem

Assume that the investigated body occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with the Lipschitz continuous boundary $\partial\Omega = \bar{\Gamma}_f \cup \bar{\Gamma}_u$ where Γ_f, Γ_u are open in $\partial\Omega$, mutually disjoint and $\Gamma_u \neq \emptyset$. On Γ_f, Γ_u, we prescribe the Neumann and the homogeneous Dirichlet boundary conditions, respectively.

We denote $\mathbb{R}^{3\times 3}_{sym}$ as the space of symmetric second order tensors. The biscalar product and the corresponding norm in $\mathbb{R}^{3\times 3}_{sym}$ will be denoted by $\boldsymbol{e} : \boldsymbol{\eta} = e_{ij}\eta_{ij}$ and $|\boldsymbol{e}|^2 = \boldsymbol{e} : \boldsymbol{e}$ for any $\boldsymbol{e}, \boldsymbol{\eta} \in \mathbb{R}^{3\times 3}_{sym}$, respectively. Let

$$V = \{\boldsymbol{v} \in W^{1,2}(\Omega; \mathbb{R}^3) \mid \boldsymbol{v}|_{\Gamma_u} = \boldsymbol{0}\} \quad \text{and} \quad Q = L^2(\Omega; \mathbb{R}^{3\times 3}_{sym}) \tag{1}$$

be the spaces of for displacement and stress (strain) fields with respect to the space variable $\boldsymbol{x} \in \Omega$. Suitable functional spaces with respect to the pseudo-time variable $t \in (0, T)$ can be found e.g. in [6].

We use the standard notation $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^p$, $\boldsymbol{u}$ for a stress tensor, a strain tensor, a plastic strain tensor, and a displacement vector, respectively, and assume that these unknown quantities depend on $\boldsymbol{x} \in \Omega$ and $t \in (0, T)$. Under the small strain assumption, we arrive at $\boldsymbol{\varepsilon} := \varepsilon(\boldsymbol{u})$, where $\varepsilon(\boldsymbol{v}) = \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T)$ for any $\boldsymbol{v} \in V$. Further, we consider the linear relation between the stress and the elastic strain: $\boldsymbol{\sigma} = \mathbb{C}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$, where $\mathbb{C}$ denotes the fourth order elastic tensor representing the Hooke law.

The load functional

$$\ell_t(\boldsymbol{v}) = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{v}\, d\boldsymbol{x} + \int_{\Gamma_f} \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{s}, \quad \boldsymbol{v} \in V,\ t \in (0, T),$$

consists of the volume forces $\boldsymbol{F} := \boldsymbol{F}(\boldsymbol{x}, t)$ and the surface forces $\boldsymbol{f} := \boldsymbol{f}(\boldsymbol{x}, t)$ applied on the part Γ_f of the boundary $\partial\Omega$. Then the weak formulation of the equilibrium equation reads as

$$\int_\Omega \boldsymbol{\sigma} : \varepsilon(\boldsymbol{v})\, d\boldsymbol{x} = \ell_t(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V,\ \forall t \in (0, T). \tag{2}$$

In order to introduce the plastic flow rule, we define at first the set

$$B = \{\boldsymbol{\tau} \in \mathbb{R}^{3\times 3}_{sym} \mid \varphi(\boldsymbol{\tau}) \leq 0\}, \tag{3}$$

of plastically admissible stress tensors. We let the function $\varphi\colon \mathbb{R}^{3\times 3}_{sym} \to \mathbb{R}$ in an abstract form and assume that φ is convex and satisfies $\varphi(\boldsymbol{0}) < 0$. Using the principle of maximum plastic dissipation, the plastic flow rule reads as

$$\dot{\boldsymbol{\varepsilon}^p} : (\boldsymbol{\tau} - \boldsymbol{\sigma}) \leq 0 \quad \forall \boldsymbol{\tau} \in B, \ \forall t \in (0, T). \tag{4}$$

Notice that the flow rule can also be defined in literature by different ways. For example, one can use the Karush-Kuhn-Tucker conditions and write [5, 12, 13]:

$$\dot{\boldsymbol{\varepsilon}^p} \in \dot{\lambda} \partial \varphi(\boldsymbol{\sigma}), \quad \dot{\lambda} \geq 0, \ \ \varphi(\boldsymbol{\sigma}) \leq 0, \ \ \dot{\lambda} \varphi(\boldsymbol{\sigma}) = 0,$$

where $\partial \varphi(\boldsymbol{\sigma})$ denotes the subdifferential of φ at $\boldsymbol{\sigma}$ and $\dot{\lambda}$ is the plastic multiplier. Making use of the convex duality, one can also write the flow rule as follows [6]:

$$-\boldsymbol{\sigma} : (\boldsymbol{q} - \dot{\boldsymbol{\varepsilon}^p}) + I_B^*(\boldsymbol{q}) - I_B^*(\dot{\boldsymbol{\varepsilon}^p}) \geq 0 \quad \forall q \in \mathbb{R}^{3 \times 3}, \tag{5}$$

where

$$I_B^*(\boldsymbol{q}) := \sup_{\boldsymbol{\tau} \in B} \boldsymbol{\tau} : \boldsymbol{q} \tag{6}$$

denotes the dissipation potential and simultaneously, the dual function to the indicator function I_B of the set B.

To complete the model, we consider the following initial conditions:

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \ \boldsymbol{\varepsilon}^p(0) = \boldsymbol{\varepsilon}_0^p, \ \ \boldsymbol{\varepsilon}(0) = \varepsilon(\boldsymbol{u}_0), \ \ \boldsymbol{\sigma}(0) = \mathbb{C}(\varepsilon(\boldsymbol{u}_0) - \boldsymbol{\varepsilon}_0^p) \quad \text{in } \Omega. \tag{7}$$

The elastoplastic problem in terms of stresses leads to the following evolution variational inequality:

$$\left\{ \begin{array}{l} \quad \text{find } \boldsymbol{\sigma} := \boldsymbol{\sigma}(t) \in \Lambda_t \cap P : \quad \int_\Omega \mathbb{C}^{-1} \dot{\boldsymbol{\sigma}} : (\boldsymbol{\tau} - \boldsymbol{\sigma}) \, d\boldsymbol{x} \geq 0 \quad \forall \boldsymbol{\tau} \in \Lambda_t \cap P, \ \ \forall t \in (0, T), \\ \Lambda_t = \{ \boldsymbol{\tau} \in Q \mid \int_\Omega \boldsymbol{\tau} : \varepsilon(\boldsymbol{v}) \, d\boldsymbol{x} = \ell_t(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V \}, \ \ P = \{ \boldsymbol{\tau} \in Q \mid \boldsymbol{\tau}(\boldsymbol{x}) \in B, \ \forall \boldsymbol{x} \in \Omega \}. \end{array} \right. \tag{8}$$

Notice that problem (8) can be derived inserting $\dot{\boldsymbol{\varepsilon}}^p = \mathbb{C}^{-1} \dot{\boldsymbol{\sigma}} - \varepsilon(\dot{\boldsymbol{u}})$ to the flow rule (4), integrating over Ω and using the definition of Λ_t. One can analyze existence and uniqueness of the solution to (8) under the assumption on the *save load condition* [6]:

$$\forall t \in (0, T) \ \exists \boldsymbol{\tau} \in \Lambda_t \cap P. \tag{9}$$

As we will see, the verification of (9) is closely related to the limit load analysis.

The problem can also be formulated in terms of displacements and plastic strains by using the dual flow rule (5) and inserting $\boldsymbol{\sigma} = \mathbb{C}(\varepsilon(\boldsymbol{u}) - \boldsymbol{\varepsilon}^p)$ to (2) and (5):

$$\int_\Omega \mathbb{C}(\varepsilon(\boldsymbol{u}) - \boldsymbol{\varepsilon}^p) : [\varepsilon(\boldsymbol{v} - \dot{\boldsymbol{u}}) - (\boldsymbol{q} - \dot{\boldsymbol{\varepsilon}^p})] \, d\boldsymbol{x} + \int_\Omega I_B^*(\boldsymbol{q}) \, d\boldsymbol{x} - \int_\Omega I_B^*(\dot{\boldsymbol{\varepsilon}^p}) \, d\boldsymbol{x} \geq \ell_t(\boldsymbol{v} - \dot{\boldsymbol{u}}), \tag{10}$$

for any $(\boldsymbol{v}, \boldsymbol{q}) \in V \times Q$ and any $t \in (0, T)$. It is well-known that (10) is a dual problem to (8) and its solvability cannot be studied on the Sobolev space V but on BD-spaces with bounded deformations [16] similarly as other problems formulated below in terms of displacements.

3 Implicit discretization of the quasistatic problem

As we shall see, the limit analysis problem presented in Section 4 is independent of the pseudo-time parameter t. Despite this fact, we introduce a time discretization of the problem since it will be useful for an interpretation a penalty method studied in Section 5.

Consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the interval [0,T] and approximate the plastic strain rate by the implicit Euler scheme:

$$\dot{\boldsymbol{\varepsilon}}^p(t_k) \approx \frac{\boldsymbol{\varepsilon}_k^p - \boldsymbol{\varepsilon}_{k-1}^p}{t_k - t_{k-1}}, \quad \boldsymbol{\varepsilon}_k^p := \boldsymbol{\varepsilon}^p(t_k), \quad k = 1, 2, \ldots, N.$$

It is well-known that discrete counterparts of problems (8) and (10) can be arranged as the following minimization problems, respectively [6]:

$$\left\{ \begin{array}{c} \text{given } \boldsymbol{\sigma}_{k-1} \in Q, \text{ find } \boldsymbol{\sigma}_k \in \Lambda_k \cap P: \quad \mathcal{J}_k^*(\boldsymbol{\sigma}_k) \le \mathcal{J}_k^*(\boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \Lambda_k \cap P, \\ \mathcal{J}_k^*(\boldsymbol{\tau}) = \frac{1}{2}\int_\Omega \mathbb{C}^{-1}\boldsymbol{\tau} : \boldsymbol{\tau}\, d\boldsymbol{x} - \int_\Omega \mathbb{C}^{-1}\boldsymbol{\sigma}_{k-1} : \boldsymbol{\tau}\, d\boldsymbol{x}, \quad \Lambda_k \equiv \Lambda_{t_k}, \end{array} \right. \tag{11}$$

$$\left\{ \begin{array}{c} \text{given } \boldsymbol{\varepsilon}_{k-1}^p \in Q, \text{ find } (\boldsymbol{u}_k, \boldsymbol{\varepsilon}_k^p) \in V \times Q: \quad \mathcal{I}_k(\boldsymbol{u}_k, \boldsymbol{\varepsilon}_k^p) \le \mathcal{I}_k(\boldsymbol{v}, \boldsymbol{q}) \quad \forall (\boldsymbol{v}, \boldsymbol{q}) \in V \times Q, \\ \mathcal{I}_k(\boldsymbol{v}, \boldsymbol{q}) = \frac{1}{2}\int_\Omega \mathbb{C}(\varepsilon(\boldsymbol{v}) - \boldsymbol{q}) : (\varepsilon(\boldsymbol{v}) - \boldsymbol{q})\, d\boldsymbol{x} + \int_\Omega I_B^*(\boldsymbol{q} - \boldsymbol{\varepsilon}_{k-1}^p)\, d\boldsymbol{x} - \ell_k(\boldsymbol{v}). \end{array} \right. \tag{12}$$

Unlike the evolution problem, the discretized problem can also be formulated only in terms of displacements. To this end, we introduce the mapping $\Pi_B : \mathbb{R}^{3\times3}_{sym} \to B$ such that

$$\Pi_B : \boldsymbol{e} \mapsto \boldsymbol{\sigma}, \quad (\boldsymbol{e} - \mathbb{C}^{-1}\boldsymbol{\sigma}) : (\boldsymbol{\tau} - \boldsymbol{\sigma}) \le 0 \quad \forall \boldsymbol{\tau} \in B. \tag{13}$$

It is easy to see that $\Pi_B(\boldsymbol{e})$ represents the closest projection of $\mathbb{C}\boldsymbol{e}$ onto B with respect to the scalar product $\mathbb{C}^{-1}\boldsymbol{\tau} : \boldsymbol{e}$ in $\mathbb{R}^{3\times3}_{sym}$. Comparing (13) with (4), the discrete flow rule can be written as follows:

$$\boldsymbol{\sigma}_k = \Pi_B(\varepsilon(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_{k-1}^p). \tag{14}$$

Inserting (14) to the equilibrium equation (2), we arrive at the following problem:

$$\text{given } \boldsymbol{\varepsilon}_{k-1}^p \in Q, \text{ find } \boldsymbol{u}_k \in V: \quad \int_\Omega \Pi_B(\varepsilon(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_{k-1}^p) : \varepsilon(\boldsymbol{v})\, d\boldsymbol{x} = \ell_k(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V. \tag{15}$$

Further, it is well-known that there exists the potential to Π_B (see, e.g., [11]):

$$\Psi_B(\boldsymbol{e}) = \sup_{\boldsymbol{\tau} \in B} \{\boldsymbol{\tau} : \boldsymbol{e} - \frac{1}{2}\mathbb{C}^{-1}\boldsymbol{\tau} : \boldsymbol{\tau}\}, \quad \boldsymbol{e} \in \mathbb{R}^{3\times3}_{sym}, \tag{16}$$

i.e., $\partial\Psi_B(\boldsymbol{e})/\partial\boldsymbol{e} = \Pi_B(\boldsymbol{e})$. Hence, problem (17) can be equivalently rewritten as the minimization problem

$$\mathcal{J}_k(\boldsymbol{u}_k) \le \mathcal{J}_k(\boldsymbol{v}) \;\; \forall \boldsymbol{v} \in V, \quad \mathcal{J}_k(\boldsymbol{v}) = \int_\Omega \Psi_B(\varepsilon(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_{k-1}^p)\, d\boldsymbol{x} - \ell_k(\boldsymbol{v}). \tag{17}$$

Notice that the functionals $\mathcal{I}_k$ and $\mathcal{J}_k$ are related as follows:

$$\min_{\boldsymbol{q} \in Q} \mathcal{I}_k(\boldsymbol{v}, \boldsymbol{q}) = \mathcal{J}_k(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V, \; \forall k = 0, 1, 2, \ldots$$

4 Limit load analysis

From now on, we consider the load functional in the form

$$\ell_t(\boldsymbol{v}) = L_0(\boldsymbol{v}) + tL(\boldsymbol{v}), \quad \boldsymbol{v} \in V, \tag{18}$$

where the functionals L_0 and L_1 are independent of t and satisfy:

$$\exists \boldsymbol{\tau}_0 \in P: \quad \int_\Omega \boldsymbol{\tau}_0 : \varepsilon(\boldsymbol{v})\, d\boldsymbol{x} = L_0(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V, \tag{19}$$

$$\exists \hat{\boldsymbol{v}} \in V: \quad L(\hat{\boldsymbol{v}}) \neq 0. \tag{20}$$

Under the assumption (19), the following implication holds:

$$\text{if} \quad \Lambda_T \cap P \neq \emptyset \quad \text{then} \quad \Lambda_t \cap P \neq \emptyset \;\; \forall t \in (0, T). \tag{21}$$

Hence, $\Lambda_T \cap P \neq \emptyset$ is a sufficient condition for the save load (9).

Within the limit analysis, the fixed value T is not prescribed. Instead of this, the limit value t^* of the parameter t is searched:

$$t^* = \sup\{t \geq 0 \mid \Lambda_t \cap P \neq \emptyset\} = \sup_{\boldsymbol{\tau} \in P} \inf_{\substack{\boldsymbol{v} \in V \\ L(\boldsymbol{v})=1}} \left\{ \int_\Omega \boldsymbol{\tau} : \varepsilon(\boldsymbol{v})\, d\boldsymbol{x} - L_0(\boldsymbol{v}) \right\}. \tag{22}$$

The problem (22) is known as the static principle of the limit analysis. The kinematic principle is dual to the static one and leads to the following minimization problem:

$$\bar{t} = \inf_{\substack{\boldsymbol{v} \in V \\ L(\boldsymbol{v})=1}} \sup_{\boldsymbol{\tau} \in P} \left\{ \int_\Omega \boldsymbol{\tau} : \varepsilon(\boldsymbol{v})\, d\boldsymbol{x} - L_0(\boldsymbol{v}) \right\} = \inf_{\substack{\boldsymbol{v} \in V \\ L(\boldsymbol{v})=1}} \left\{ \int_\Omega I_B^*(\varepsilon(\boldsymbol{v}))\, d\boldsymbol{x} - L_0(\boldsymbol{v}) \right\}, \tag{23}$$

where I_B^* is defined by (6). From the duality, it follows that

$$t^* \leq \bar{t}, \tag{24}$$

i.e., $\bar{t}$ is an upper bound of t^*. Nevertheless, the equality in (24) was shown for bounded sets B (see [8]) and some unbounded ones representing, e.g., by the von Mises, Tresca [16, 4] or Drucker-Prager yield criteria [9]. Further, it is readily seen that problems (22) and (23) are independent of the time variable and thus the same problems can also be introduced for the discretized problem defined in Section 3 or for the generalized Hencky problem [16, 4, 7, 8]. The limit analysis problems are also independent of the elastic tensor $\mathbb{C}$ and describe the collapse state of the body.

The kinematic limit analysis problem defined by (23) contains the linear equality constraint on the load L and other eventual constraints depending on the set B as follows from the definition of I_B^*. We introduce two examples of B for illustration (see, e.g., [8, 14]).

1. If the von Mises yield criterion is considered then $B = \left\{\boldsymbol{\tau} \in \mathbb{R}^{3\times 3}_{sym} \mid |\boldsymbol{\tau}^D| \leq \gamma\right\}$, where $\boldsymbol{\tau}^D$ denotes the deviatoric part of $\boldsymbol{\tau}$ and $\gamma > 0$ represents the initial yield stress. The related limit analysis problem reads:

$$\bar{t} = \inf_{\substack{\boldsymbol{v}\in V,\ \mathrm{div}\,\boldsymbol{v}=0\\ L(\boldsymbol{v})=1}} \left\{\int_\Omega \gamma|\varepsilon(\boldsymbol{v})|\,d\boldsymbol{x} - L_0(\boldsymbol{v})\right\}, \quad \mathrm{div}\,\boldsymbol{v} = \mathrm{trace}\,\varepsilon(\boldsymbol{v}). \tag{25}$$

2. If the Drucker-Prager yield criterion is considered then

$$B = \left\{\boldsymbol{\tau} \in \mathbb{R}^{3\times 3}_{sym} \mid \frac{a}{3}\,\mathrm{trace}\,\boldsymbol{\tau} + |\boldsymbol{\tau}^D| \leq \gamma\right\}, \quad a, \gamma > 0.$$

   The related limit analysis problem reads:

$$\bar{t} = \inf_{\substack{\boldsymbol{v}\in V,\ L(\boldsymbol{v})=1\\ \mathrm{div}\,\boldsymbol{v}\geq a|\varepsilon^D(\boldsymbol{v})|}} \left\{\int_\Omega \frac{\gamma}{a}\,\mathrm{div}\,\boldsymbol{v}\,d\boldsymbol{x} - L_0(\boldsymbol{v})\right\}. \tag{26}$$

5 Penalization of the kinematic limit analysis problem

We have illustrated that the kinematic limit analysis problem can contain very difficult constraints at each point of Ω causing locking phenomena. To eliminate these constraints, we introduce two possible ways of penalization to the problem (23).

5.1 Truncation method

The first penalization is based on replacing unbounded B by its bounded, convex subset B_m and thus can be interpreted as the truncation method. Notice that the function $I^*_{B_m}$ is real-valued for any $\boldsymbol{\tau} \in \mathbb{R}^{3\times 3}_{sym}$ and thus the penalized problem

$$\bar{t}_m = \inf_{\substack{\boldsymbol{v}\in V\\ L(\boldsymbol{v})=1}} \left\{\int_\Omega I^*_{B_m}(\varepsilon(\boldsymbol{v}))\,d\boldsymbol{x} - L_0(\boldsymbol{v})\right\}, \tag{27}$$

contains only the basic constraint on L. Let t^*_m denote the static limit load parameter from (22), where B_m is used instead of B. Then the following relations hold [8, 14]:

$$t^*_m = \bar{t}_m \leq t^* \leq \bar{t}. \tag{28}$$

We see that the penalized limit load parameters $\bar{t}_m$ and t^*_m coincide and that they are lower bounds of t^* and $\bar{t}$.

For the bounded set B_m, we have also stronger convergence results with respect to the (space) discretization parameter than for unbounded B, see [7, 8, 14]. Denote V_h as a finite element approximation of V and assume that the system $\{V_h\}_h$ is limit dense in V. Then $\bar{t}_{m,h} \to \bar{t}_m$ as $h \to 0_+$, where $\bar{t}_{m,h}$ denotes the the discrete limit load parameter obtained by the finite element approximation of problem (27).

The discrete counterpart to problem (27) can be solved, e.g., by the indirect incremental method presented below. We refer to [7, 8, 14] for some illustrative numerical examples.

5.2 Indirect incremental method

The indirect method of incremental limit analysis was originally introduced in [15, 2] for the discretized Hencky problem containing the von Mises yield criterion. Their extension for continuous setting of the Hencky problem and an abstract yield criterion was done in [7, 8]. This method was interpreted as the penalization method to the limit analysis problem in [14]. Here, we generalize the method for $L_0 \neq 0$ and relate it to the problem (17).

This penalization is based on the following relations between the functions I_B^* and Ψ_B:

$$\lim_{\alpha\to+\infty} \frac{1}{\alpha}\Psi_B(\alpha\boldsymbol{e}-\boldsymbol{\eta}) = I_B^*(\boldsymbol{e}) \quad \forall \boldsymbol{e},\boldsymbol{\eta} \in \mathbb{R}^{3\times 3}_{sym}, \quad \frac{1}{\alpha}\Psi_B(\alpha\boldsymbol{e}) \leq I_B^*(\boldsymbol{e}) \quad \forall \alpha > 0. \tag{29}$$

To be in accordance with Section 3, we choose $\boldsymbol{\eta} = \boldsymbol{\varepsilon}^p_{k-1}$ in (29) and define the following penalization of problem (23):

$$\left\{\begin{array}{c} \text{given } \alpha > 0,\ \boldsymbol{\varepsilon}^p_{k-1} \in Q,\ \text{find } \boldsymbol{u}_k^\alpha \in V,\ L(\boldsymbol{u}_k^\alpha) = 1 : \\ \mathcal{J}_k^\alpha(\boldsymbol{u}_k^\alpha) \leq \mathcal{J}_k^\alpha(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V,\ L(\boldsymbol{v}) = 1, \\ \mathcal{J}_k^\alpha(\boldsymbol{v}) = \int_\Omega \frac{1}{\alpha}\Psi_B(\alpha\boldsymbol{e} - \boldsymbol{\varepsilon}^p_{k-1})\, d\boldsymbol{x} - L_0(\boldsymbol{v}). \end{array}\right. \tag{30}$$

Enforcing the constraint $L(\boldsymbol{v}) = 1$ by a Lagrange multiplier and using the differentiability of Ψ_B, we arrive from (30) at the following saddle point system:

$$\left\{\begin{array}{c} \text{given } \alpha > 0,\ \boldsymbol{\varepsilon}^p_{k-1} \in Q,\ \text{find } t_k := t_k(\alpha),\ \boldsymbol{u}_k^\alpha \in V : \\ \int_\Omega \Pi_B(\varepsilon(\alpha\boldsymbol{u}_k^\alpha) - \boldsymbol{\varepsilon}^p_{k-1}) : \varepsilon(\boldsymbol{v})\, d\boldsymbol{x} = L_0(\boldsymbol{v}) + t_k L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V, \\ L(\boldsymbol{u}_k^\alpha) = 1. \end{array}\right. \tag{31}$$

Recalling $\ell_k = L_0 + t_k L$ and comparing (31) with (17), we observe that the following statements hold. If $(t_k, \boldsymbol{u}_k^\alpha) \in \mathbb{R}_+ \times V$ is a solution to (31) then $\boldsymbol{u}_k = \alpha\boldsymbol{u}_k^\alpha$ solves (17) for $t_k := t_k(\alpha)$. Conversely, if $\boldsymbol{u}_k \in \mathbb{V}$ is a solution to (17) satisfying $L(\boldsymbol{u}_k) > 0$ for given $t_k > 0$ then $(t_k, \boldsymbol{u}_k^\alpha)$, where $\boldsymbol{u}_k^\alpha = \boldsymbol{u}_k/L(\boldsymbol{u}_k)$, solves (31) for $\alpha = L(\boldsymbol{u}_k)$.

The direct method of incremental limit analysis is based on an adaptive construction of the sequence

$$0 < t_0 < t_1 < \ldots < t_k < \ldots < t^*$$

depending on the solvability of problem (17). Within the indirect method, an unbounded sequence

$$\alpha_0 < \alpha_1 < \ldots < \alpha_k < \ldots$$

is constructed and the corresponding sequence $\{t_k\}$ of solutions to (31) is computed. One can expect that the sequence $\{t_k\}$ is nondecreasing and tending to t^* as $k \to +\infty$. This was shown in [7] under the following simplified assumptions:

$$L_0 = 0 \quad \text{and} \quad \boldsymbol{\varepsilon}^p_{k-1} = \boldsymbol{0}, \quad k = 1, 2, \ldots. \tag{32}$$

Moreover, (32) and (21) imply that the mapping $\alpha \mapsto t_k(\alpha) \equiv t(\alpha)$ is uniquely defined, continuous and $t(\alpha) \to 0$ as $\alpha \to 0$ from above [7].

As above, consider the system $\{V_h\}_h$ of finite element approximations of the space V. Then the discrete counterpart $t_h := t_h(\alpha)$ has analogous properties as $t := t(\alpha)$. Moreover, the following pointwise convergence holds for any $\alpha > 0$: $t_h(\alpha) \to t(\alpha)$ as $h \to 0_+$.

In order to extend these results to nontrivial L_0, it seems to be sufficient to assume that there exists $\boldsymbol{u}_0 \in \mathbb{V}$ satisfying

$$\int_\Omega \Pi_B(\varepsilon(\boldsymbol{u}_0)) : \varepsilon(\boldsymbol{v})\, d\boldsymbol{x} = L_0(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V \quad \text{and} \quad L(\boldsymbol{u}_0) \geq 0. \tag{33}$$

Notice that $(33)_1$ implies the assumption (19) while $(33)_2$, ensures $t(\alpha) \to 0$ as $\alpha \to 0$.

6 Newton-like method for the indirect incremental method

The finite element approximation of problem (31) leads to the following algebraic system:

$$\text{given } \alpha_k > 0, \text{ find } (\boldsymbol{u}_k, t_k) \in \mathbb{R}^n \times \mathbb{R}_+ : \quad \left\{ \begin{array}{c} \boldsymbol{F}_k(\boldsymbol{u}_k) = \boldsymbol{l}_0 + t_k \boldsymbol{l}, \\ \boldsymbol{l}^T \boldsymbol{u}_k = 1. \end{array} \right. \tag{34}$$

Notice that the nonlinear function $\boldsymbol{F}_k \colon \mathbb{R}^n \to \mathbb{R}^n$ is assembled using the operators Π_B at each integration point and depends on the solution from the previous step t_{k-1} and the given value α_k. Since Π_B is not smooth everywhere, the same also holds for $\boldsymbol{F}_k$. On the other hand, one can study the semismoothness of Π_B or $\boldsymbol{F}_k$, and introduce generalized derivatives of these functions [11, 12, 13]. The generalized derivative of Π_B is known as the consistent tangent operator in literature. Using this operator, one can assemble the generalized derivative of $\boldsymbol{F}_k$, which is represented by a mapping $\boldsymbol{K}_k : \mathbb{R}^n \to \mathbb{R}^{n \times n}$.

A nonsmooth (or semismooth) version of the Newton method to problem (34) leads to the following algorithm:

Algorithm 1 (ALG-α)

1: initialization: $\boldsymbol{u}_k^0$, t_k^0
2: **for** $i = 0, 1, 2, \ldots$ **do**
3: find $\boldsymbol{v}^i$, $\boldsymbol{w}^i \in \boldsymbol{V}$: $\boldsymbol{K}_k(\boldsymbol{u}_k^i)\boldsymbol{v}^i = \boldsymbol{l}_0 + t_k^i \boldsymbol{l} - \boldsymbol{F}_k(\boldsymbol{u}_k^i)$, $\boldsymbol{K}_k(\boldsymbol{u}_k^i)\boldsymbol{w}^i = \boldsymbol{l}$
4: compute $\delta t^i = [1 - \boldsymbol{l}^T(\boldsymbol{u}_k^i + \boldsymbol{v}^i)]/\boldsymbol{l}^T \boldsymbol{w}^i$
5: compute $\delta \boldsymbol{u}^i = \boldsymbol{v}^i + \delta t^i \boldsymbol{w}^i$
6: set $\boldsymbol{u}_k^{i+1} = \boldsymbol{u}_k^i + \delta \boldsymbol{u}^i$, $t_k^{i+1} = t_k^i + \delta t^i$
7: **if** $\|\delta \boldsymbol{u}^i\|/(\|\boldsymbol{u}_k^{i+1}\| + \|\boldsymbol{u}_k^i\|) \leq \epsilon_{Newton}$ **then stop**
8: **end for**
9: set $\boldsymbol{u}_k = \boldsymbol{u}_k^{i+1}$, $t_k = t_k^{i+1}$.

Further, we initialize ALG-α using the linear extrapolation of the solutions from two previous steps $k-2$ and $k-1$, $k \geq 2$ [13]:

$$\boldsymbol{u}_k^0 = \boldsymbol{u}_{k-1} + \frac{\alpha_k - \alpha_{k-1}}{\alpha_{k-1} - \alpha_{k-2}}(\boldsymbol{u}_{k-1} - \boldsymbol{u}_{k-2}), \quad t_k^0 = t_{k-1} + \frac{\alpha_k - \alpha_{k-1}}{\alpha_{k-1} - \alpha_{k-2}}(t_{k-1} - t_{k-2}).$$

We observe that this initialization is more convenient than $\boldsymbol{u}_k^0 = \boldsymbol{u}_{k-1}$, $t_k^0 = t_{k-1}$.

Local superlinear convergence of ALG-α was analyzed in [2]. There was also proposed some modifications of the algorithm in order to receive global convergence results.

We refer to [2, 7, 8, 14, 13] for some illustrative numerical examples on the indirect method of the incremental limit analysis. For unbounded B, we observe that it is more convenient to use higher order finite elements. In the case of $P1$ or $Q1$ elements, we recommend to combine the indirect incremental method with the truncation method to reduce expected locking phenomena.

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REFERENCES

[1] Caboussat, A. and Glowinski, R. Numerical solution of a variational problem arising in stress analysis: the vector case. *Discrete Contin. Dyn. Syst* (2010) **27**: 1447–1472.

[2] Cermak, M., Haslinger, J., Kozubek, T., Sysala, S. Discretization and numerical realization of contact problems for elastic-perfectly plastic bodies. PART II - numerical realization, limit analysis. *ZAMM - Z. Angew. Math. Mech.* (2015) **95**: 1348–1371.

[3] Chen, W.F. and Liu, X. *Limit analysis in soil mechanics*. Elsevier, 2012.

[4] Christiansen, E. Limit analysis of collapse states, in: *Handbook of numerical analysis* (P.G. Ciarlet and J.L. Lions eds., Vol IV, Part 2), North-Holland, 1996, pp. 193–312.

[5] de Souza Neto, E. A., Perić, D., Owen, D. R. J.: *Computational methods for plasticity: theory and application*. Wiley, (2008).

[6] Han W, Reddy BD. *Plasticity: mathematical theory and numerical analysis*. Springer, 1999.

[7] Haslinger, J., Repin, S., Sysala S. A reliable incremental method of computing the limit load in deformation plasticity based on compliance: Continuous and discrete setting. *Journal of Computational and Applied Mathematics* (2016) **303**: 156–170.

[8] Haslinger, J., Repin, S., Sysala S. Guaranteed and computable bounds of the limit load for variational problems with linear growth energy functionals. *Applications of Mathematics* (2016) **61**: 527–564.

[9] Repin, S. and Seregin, G. Existence of a weak solution of the minimax problem arising in Coulomb-Mohr plasticity. *Nonlinear evolution equations* (N.N. Uraltseva, ed.), *Translations of the American Mathematical Society*, Series 2 (1995) **164**: 189–220.

[10] Sloan, S. Geotechnical stability analysis, *Géotechnique* (2013) **63**, 531–572.

[11] Sysala, S. Properties and simplifications of constitutive time-discretized elastoplastic operators. *ZAMM - Z. Angew. Math. Mech.* (2014) **94**: 233–255.

[12] Sysala, S., Cermak, M., Koudelka, T., Kruis, J., Zeman, J., Blaheta, R. Subdifferential-based implicit return-mapping operators in computational plasticity. *ZAMM - Z. Angew. Math. Mech.* (2016) **96**: 1318–1338.

[13] Sysala, S., Cermak, M., Ligurský, T. Subdifferential-based implicit return-mapping operators in Mohr-Coulomb plasticity. (Submitted in 2017), http://arxiv.org/abs/1508.07435.

[14] Sysala, S. and Haslinger, J. Truncation and indirect incremental methods in Hencky's perfect plasticity, in: *Mathematical Modelling in Solid Mechanics*, Springer, 2017, pp. 265–284.

[15] Sysala, S., Haslinger, J., Hlaváček, I., Cermak, M. Discretization and numerical realization of contact problems for elastic-perfectly plastic bodies. PART I - discretization, limit analysis. *ZAMM - Z. Angew. Math. Mech.* (2015) **95**: 333–353.

[16] Temam, R.: *Mathematical Problems in Plasticity.* Gauthier-Villars, Paris, (1985).

[17] N. Zouain, I. Pontes Filho, L. Borges, and L. M. da Costa, Plastic collapse in non-associated hardening materials with application to cam-clay. *International journal of solids and structures* (2007) **44**: 4382–4398.